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Lax Representation and Kronecker Product

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We show how the Kronecker product can be used to find new Lax representations.

A number of interesting dynamical systems can be written in Lax representation (Toda, 1989; Van Moerbeke, 1986; Steeb and Euler, 1988; Steeb, 1994a; Steeb and van Tonder, 1988)

$$\frac{dL}{dt} = [A, L](t) \tag{1}$$

where A and L are given by $n \times n$ matrices. An example is the Toda lattice. Given two Lax representations, we show how the Kronecker product can be used to find a new Lax representation. We then give an application. Finally, we discuss some extensions.

The Kronecker product (Steeb, 1991, 1994b; Steeb and Lewien, 1992) plays an important role in the fast Fourier transform, Hadamard transform, statistical physics, quantum groups, etc. Let X and R be $m \times m$ matrices and Y and P be $n \times n$ matrices. Then we have

$$(X \otimes Y)(R \otimes P) \equiv (XR) \otimes (YP) \tag{2}$$

Let us consider two Lax representations

$$\frac{dL}{dt} = [A, L](t), \qquad \frac{dM}{dt} = [B, M](t) \tag{3}$$

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where L and A are $m \times m$ matrices and M and B are $n \times n$ matrices. Let I_m be the $m \times m$ unit matrix and I_n be the $n \times n$ unit matrix. Then we find the Lax representation

$$\frac{d}{dt}(L \otimes M) = [A \otimes I_n + I_m \otimes B, L \otimes M](t) \tag{4}$$

This can easily be seen as follows: The right-hand side of (4) is given by

$$[A \otimes I_n + I_m \otimes B, L \otimes M] = [A \otimes I_n, L \otimes M] + [I_m \otimes B, L \otimes M] \tag{5}$$

Thus using (2) we find that

$$[A \otimes I_n + I_n \otimes B, L \otimes M]$$
$$= (AL) \otimes M - (LA) \otimes M + L \otimes (BM) - L \otimes (MB) \tag{6}$$

On the other hand, we have

$$\frac{d}{dt}(L \otimes M) = \frac{dL}{dt} \otimes M + L \otimes \frac{dM}{dt} \tag{7}$$

Inserting (3) into (7), we find the right-hand side of (6). Thus (4) holds.

From (1) we find that the first integrals are given by

$$F_k = \mathrm{tr}(L^k), \qquad k = 1, 2, \ldots \tag{8}$$

Since

$$\mathrm{tr}(X^k \otimes Y^j) = \mathrm{tr}(X^k)\,\mathrm{tr}(Y^j) \tag{9}$$

where X is an $m \times m$ matrix and Y is an $n \times n$ matrix, we find that the first integrals of (4) are given by

$$F_{kj} = \mathrm{tr}(L^k)\,\mathrm{tr}(M^j), \qquad k, j = 1, 2, \ldots \tag{10}$$

Obviously, we can extend this to more than two Lax pairs. For example, given three Lax representations

$$\frac{dL}{dt} = [A, L](t), \qquad \frac{dM}{dt} = [B, M](t), \qquad \frac{dN}{dt} = [C, N](t) \tag{11}$$

we find that

$$\frac{d}{dt}(L \otimes M \otimes N)$$
$$= [A \otimes I_n \otimes I_p + I_m \otimes B \otimes I_p + I_m \otimes I_n \otimes C, L \otimes M \otimes N](t) \tag{12}$$

where C and N are $p \times p$ matrices.

As an example, consider the nonlinear system of ordinary differential equations

$$\frac{du_1}{dt} = (\lambda_3 - \lambda_2)u_2u_3$$
$$\frac{du_2}{dt} = (\lambda_1 - \lambda_3)u_3u_1 \tag{13}$$
$$\frac{du_3}{dt} = (\lambda_2 - \lambda_1)u_1u_2$$

where $\lambda_j \in \mathfrak{R}$. This system describes Euler's rigid-body motion. The first integrals are given by

$$I_1(u) = u_1^2 + u_2^2 + u_3^2, \qquad I_2(u) = \lambda_1u_1^2 + \lambda_2u_2^2 + \lambda_3u_3^2 \tag{14}$$

A Lax representation is given by

$$\frac{dL}{dt} = [L, \lambda L](t) \tag{15}$$

where

$$L := \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}, \qquad \lambda L := \begin{pmatrix} 0 & -\lambda_3u_3 & \lambda_2u_2 \\ \lambda_3u_3 & 0 & -\lambda_1u_1 \\ -\lambda_2u_2 & \lambda_1u_1 & 0 \end{pmatrix} \tag{16}$$

Then $\mathrm{tr}(L)^k$ $(k = 1, 2, \ldots)$ provides only one first integral. We obtain

$$\mathrm{tr}\, L = 0, \qquad \mathrm{tr}\, L^2 = -2(u_1^2 + u_2^2 + u_3^2) = -2I_1 \tag{17}$$

Since L does not depend on λ, we cannot find I_2. This Lax representation can be applied to (3) when we set $M = L$.

To overcome the problem of finding only one first integral of (13) we consider now (Van Moerbeke, 1986; Steeb and van Tonder, 1988)

$$\frac{d(L + Ay)}{dt} = [L + Ay, \lambda L + By](t) \tag{18}$$

where y is a dummy variable and A and B are time-independent diagonal matrices, i.e., $A = \mathrm{diag}(A_1, A_2, A_3)$ and $B = \mathrm{diag}(B_1, B_2, B_3)$ with $A_j, B_j \in \mathfrak{R}$. Equation (18) decomposes into various powers of y, namely

$$y^0\!: \quad \frac{dL}{dt} = [L, \lambda L]$$
$$y^1\!: \quad 0 = [L, B] + [A, \lambda L]$$
$$y^2\!: \quad [A, B] = 0$$

The last equation is satisfied identically since A and B are diagonal matrices. The second equation leads to

$$\lambda_i = \frac{B_j - B_k}{A_j - A_k} \tag{19}$$

where (i, j, k) are permutations of (1, 2, 3). It can be satisfied by setting

$$B_j = A_j^2 \tag{20}$$

and

$$\lambda_i = A_j + A_k \tag{21}$$

Consequently the original Lax pair L, λL satisfies the extended Lax pair

$$L + Ay, \quad \lambda L + By \tag{22}$$

Now $\mathrm{tr}[(L + Ay)^2]$ and $\mathrm{tr}[(L + Ay)^3]$ provide the first integrals (14). For this extended Lax pair the concept of the Kronecker product described above can also be applied.

For some dynamical systems such as the energy level motion (Yukawa, 1986; Steeb and Louw, 1988) we find an extended Lax representation

$$\frac{dL}{dt} = [A, L](t), \qquad \frac{dK}{dt} = [A, K](t) \tag{23}$$

where L and K do not commute. For this extended system of Lax representations we can also apply the Kronecker product technique given above in order to find new Lax representations.

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